Research in Mathematics Education

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Submission ID	240071914
Article Type	Research Article
Keywords	Factored forms, concept definitions, concept ima ges
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Students' conceptions of a factored form through concept definitions and concept images

This research explored third-year students' understanding of the definition of a factored form, the use of concept definitions and concept images in identifying factored forms, and factorisation over different number systems. Seventeen participants filled out an online questionnaire with five open and closed questions. Their answers were analysed using descriptive statistics and thematic analysis. The research results showed that 29% of students were judged as having a clear, precise, and accurate definition; 47% defined minimally; and the rest had incomplete definitions. Students identified factored forms of algebraic expressions using their concept definitions and concept images. However, it was found that there was a conflict between their definitions and the answers students provided. Some students answered outside their definitions and gave reasons based on their cognitive structures and experiences. Although most students understood a factored form as a product of irreducible factors, they believed that the terms of powers of different elements were not crucial in factoring. Regarding irreducibility, three students consistently thought that an algebraic expression could only be factored over integers.

Keywords: factored forms; concept definitions; concept images

Introduction

Elementary algebra is a core topic in contemporary mathematics, starting in school and continuing through the study of advanced mathematics at university. Elementary algebra includes computational processes; many are reversible. An example of a reversible process is the multiplication of terms and, in reverse, splitting a single term into factors. The process of splitting a term into multiplicative factors is called factorisation. Factorisation can be applied to integers, e.g. $12 \rightarrow 2\times6$, to polynomials, e.g. $x^2-1 \rightarrow (x-1) \times (x+1)$, or indeed to terms in any algebraic ring.

Factorisation is both a computational process and a fundamental mathematical concept, especially in algebra. Procedural competence in factoring is required in many algorithms, and procedural understanding is a key step in many problem-solving techniques. For example, solving polynomial equations can be done with methods involving the factored form. A conceptual understanding of factoring will be pivotal for understanding many mathematical courses at the university, including parts of calculus, geometry, number theory, and real analysis. A conceptual understanding of factoring allows us to approach mathematical problems from different perspectives and provides a more profound understanding of many mathematical structures.

Students are very likely to learn factoring at many levels of mathematics education. Lee and Heid (2018) stated that factorisation appears regularly during school through to the university across levels of mathematics in different contexts. At the elementary mathematics level, students learn to factor in an integer, which involves breaking the number into its prime multiplicative factors. Factoring integers in practice is a foundation for the Fundamental Theorem of Arithmetic. Later, at secondary mathematics levels, factorisation of polynomials becomes an essential concept for understanding the Fundamental Theorem of Algebra. Students are taught that factorisation involves variables and coefficients, typically to solve polynomial equations and find the roots via factoring. In higher education, students may learn factorisation over a broader number system, e.g. $x^2+1 \rightarrow (x+i)\ (x-i)$ (Childs, 2009).

To grasp abstract concepts, it is essential to have clear definitions (Darmofal et al., 2002). Tennyson and Park (1980) added that a definition of a concept, called a concept definition, should be taught before providing students with examples and non-examples. Students given a concept definition performed better on understanding examples and solving problems. Mathematical definitions are a foundation for understanding a concept more formally. Concept definitions have five roles:

(1) giving a name to objects of a particular class and introducing new notations (Torkildsen et al., 2023);

(2) introducing the entities within a theory and encapsulating the essence of a concept by conveying its characteristics (Pimm, 1993);

(3) serving as a basic element for constructing a mathematical concept (Edwards & Ward, 2008; Torkildsen et al., 2023; Vinner, 1991);

(4) laying the groundwork for proofs and problem-solving (Torkildsen et al., 2023; Vinner, 1991); and

(5) fostering consistency in interpreting concepts and facilitating efficient communication of mathematical ideas (Torkildsen et al., 2023).

Thus, mathematical concept definitions become crucial for enhancing and strengthening students' understanding of a concept. However, in this context, it is interesting to note that students may never define the factored form formally. Further, they may never discuss how the definition may (or may not) differ between situations such as integer or polynomial rings.

The importance of definitions is widely acknowledged. For example, Edwards and Ward (2008) stated that students are required to have and use a definition of a concept in mathematics courses at the university. Students studying advanced mathematics are taught about definitions and must be trained to use these definitions as the main criteria in mathematics tasks (Vinner, 1991). Whether students can use formal definitions can only be established with assigned problems that require more than students' informal concept images but also formal concept definitions. However, considering the significance attributed to definitions in advanced mathematics, it is perplexing that students are found to apply their personal definitions or concept images rather than formal definitions in solving mathematical tasks related to some topics such as infinite decimal (Edwards & Ward, 2004), limit (Beynon & Zollman, 2015), and calculus (Dahl, 2017; Rösken & Rolka, 2007). Hence, we ultimately consider the profound value of formal definitions for mathematicians in other topics, including factorisation.

We investigate students' conceptions of a factored form as an interesting subject in its own right and to help us understand the role of formal mathematical definitions more generally in practical situations. The mathematical term *factored/factoring* has its mathematical meanings, independent of any natural language usage, which can help students formally differentiate the term of factoring or factored forms from other mathematical definitions. Factoring is explicitly defined in some textbooks. However, students may have definitions based on their knowledge or experience, which can differ from what the textbooks explain. The formal definition offered by textbooks may also change throughout mathematical education, from factoring integers in elementary work to abstract algebra in later university courses.

Most existing research has concentrated on the procedural use of factorisation in manipulating mathematical expressions, for example, when solving equations and determining roots, rather than studying the broader spectrum of students' factorisation definitions and their roles in advanced mathematical concepts. Zhu and Simon (1987) researched students' understanding of simplifying fractions and factoring quadratic expressions. Mok (2009) explored students' perspectives on solving polynomials. Lee and Heid (2018) studied the role of structural perspectives regarding factorisation. Therefore, a noticeable research gap exists in understanding the diverse implications of different conceptual factorisation definitions across various mathematical contexts.

The objectives of this study are to explore students' existing factorisation definitions, uncover potential relationships between students' concept definitions and concept images in identifying a factored form, investigate students' understanding of a factored form over different number systems, and identify any conflict between students' concept definitions and concept images. This study will also provide valuable insight into the role of definitions in conceptual understanding and how concept definitions may connect to conceptual questions for assessing conceptual understanding.

Mathematical background on factored forms

The factored form is both theoretically and practically important. The practical utility of the factored form arises from the fact that in an integral domain

$$A = 0 \text{ or } B = 0 \text{ if and only if } AB = 0.$$

Hence, one way to solve an equation $p = 0$ is to write p as a product, and then split the complex problem in to many simpler problems for each factor in the product.

From a theoretical perspective, the interest in factored forms arises in relation to the Fundamental Theorem of Arithmetic, and its generalisations to unique factorisation theorems in abstract algebra and number theory. The Fundamental Theorem of Arithmetic states that every integer greater than 1 can be represented as a product of prime numbers, and this representation is unique up to the order of the factors.

Research questions

Our research questions are:

(1) Do third-year undergraduate mathematics students have a formal concept definition of a factored form of a polynomial? If so, what is their definition?

(2) How do third-year undergraduate mathematics students use their concept definition and concept image for identifying a factored form?

(3) To what extent do third-year undergraduate mathematics students' concept definitions of a factored form conflict with their concept images?

(4) To what extent do third-year undergraduate mathematics students understand that factorisation takes place over different number systems?

(5) How important are powers of distinct terms in factorisation for third-year undergraduate mathematics students?

The definition of a factored form in textbooks

Many textbooks work inductively through a sequence of progressive examples and, in doing so, avoid an explicit definition of a factored form. Typical sequences of examples for factoring polynomials proceed by

(1) Taking out numerical factors.

(2) Taking out common monomial factors.

(3) Quadratic factoring with the difference of two squares as a distinguished case.

(4) Obscured/disguised quadratics, i.e. quadratics in the compound term. For example, a^4b^4-9 (Hall & Knight, 1962, p. 134).

Such a sequence can be found in older books, such as Hall and Knight (1962, Chapt XVII)1[1]. Since factoring and expanding are opposite directions of a reversible process, retaining equivalence, the topic of expanding out brackets is sometimes interleaved or included as motivating examples.

Some books do include formal definitions.

[1] This was a particularly popular algebra book, first published in 1885 and still in print in 1962, 77 years later.

> §125. DEFINITION. When an algebraic expression is the product of two or more expressions, each of these latter quantities is called a factor of it, and the determination of these quantities is called the resolution of the expression into its factors. (Hall & Knight, 1962, p. 120)

A similar definition is found in a contemporary textbook.

> An expression is factored if it is written as a product of factors.
> For example, $(x^2+1)(x-1)$ is factored but $(x+1)(x-1)-3$ is not.
> An equation is written in a factored form if one side is fully factored and the other side is zero. (Haese et al., 2019, p. 74)

Zill and Dewar (2011) mentioned that writing a polynomial as a product of other polynomials is called factoring. For instance, $3x^2$ and x^2+2 are factors of $3x^4+6x^2$ because $3x^4+6x^2 = 3x^2(x^2+2)$.

Flanders and Price (1975) explained that "factoring is the process of expressing a polynomial as a product of polynomials of lower degree (factors)" (p. 38). If a given polynomial has integer coefficients, we generally look for factors with integer coefficients.

Sangwin (2013, p. 92) stated the following definition of a factored form:

> Definition: An expression is said to be factored if it is written as a product of powers of distinct irreducible terms. The terms are known as "factors".

There are a number of separate parts to this definition.

(1) Writing an expression as a product of terms.

(2) Using powers of distinct terms, (e.g. $(x-1)^2$ rather than $(x-1)(x-1)$).

(3) Irreducibility of the individual terms.

Irreducibility means we cannot find further factors, but, we need some care here. Bradford et al. (2010) identified the following meanings, illustrated by x^8+16x^4+48.

(1) Any non-trivial factorisation, i.e. $(x^4 +4)(x^4+12)$.

(2) A factorisation into irreducible factors over the integers, i.e. $(x^2+2x+2)(x^2-2x+2)(x^4+12)$.

(3) A factorisation into terms irreducible over the reals, i.e. $(x^2+2x+2)(x^2-2x+2)(x^2+2\sqrt[4]{3}x+ 2\sqrt[4]{3})(x^2-2\sqrt[4]{3}x+2\sqrt[4]{3})$.

(4) A factorisation into irreducible polynomials over the Gaussian integers, with i allowed, i.e. $(x+1+i)(x+1-i)(x-1+i)(x-1-i)(x^4+12)$.

(5) A complete factorisation over the complex numbers, where the factor $(x^4 +12)$ would also be split into the four linear terms $x\pm\sqrt[4]{3}(1\pm i)$.

Childs (2009) also explained another example of irreducibility by x^3-2. This is a polynomial with coefficients in $\mathbb{Q}$ where $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$, so the factors:

(1) x^3-2 is irreducible over $\mathbb{Q}$

(2) $x^3-2 = (x-\sqrt[3]{2})(x^2+\sqrt[3]{2}x+\sqrt[3]{4})$ over $\mathbb{R}$

(3) $x^3-2 = (x-\sqrt[3]{2})(x+\sqrt[3]{2}((1+i\sqrt{3})/2))(x+\sqrt[3]{2}((1-i\sqrt{3})/2)$ over $\mathbb{C}$

Thus, whether a polynomial is irreducible depends on the field of coefficients. This field may be explicit, an established convention, or implicit.

Concept definitions and concept images

Vinner (1983) defined *a concept definition* as accurate verbal explanations of a concept. D. O. Tall and Vinner (1981) added that a concept definition means a set of words utilised to precisely define that concept, which can be a personal or formal definition. A concept definition can be either personal, based on an individual’s understanding or reconstruction of the concept, or formal, if it is widely accepted and recognised in a broader mathematical community. When a student can create a formal definition, it may signify a deep understanding of the concept (Vinner, 1991). The understanding is indicated by the student's ability to work with the definition to perform correctly on a problem (Edwards & Ward, 2008; Rupnow & Randazzo, 2023). Nonetheless, when the reconstruction of formal definitions is achieved only by mere memorisation, then the understanding of the concept becomes inadequate (Vinner, 1991). Although memorising a definition is no detriment, it will not foster the cognitive capacity that this definition has on the student's mathematical thinking (Vinner, 1991).

A concept image is a non-verbal association in our minds linked to the concept's name (Vinner, 1991). D. O. Tall and Vinner (1981) used a concept image to characterise the complete cognitive structure linked to the concept. The concept image encompasses all mental pictures and visual representations, associated properties, and processes. A concept image develops over time through various experiences, evolving as the individual encounters new stimuli and undergoes matures. It is evident that referencing a concept image is only applicable to a particular individual. Therefore, Vinner (1991) stated that individuals may exhibit varying reactions to a specific term (concept name) in different situations.

A concept definition links to a concept image. Bingolbali and Monaghan (2008) stated that the notion of a concept image and concept definition holds significant importance within the field of mathematics. A concept definition is occasionally taught first to help form a concept image (Vinner, 1983). After learning a concept definition, students engage with the concept, and start to form their concept images shaped by their experiences, interactions, and interpretations of the formal definition. Students have a solid conceptual understanding when the concept image aligns well with the concept definition. However, during the interaction, a potential conflict factor may occur between the concept image or concept definition and another part of the concept image or concept definition (D. O. Tall & Vinner, 1981, p. 153). Conflicts can arise from inconsistent ideas within the concept image and contradictions between the concept image and the formal definition. The situation when students are uncomfortable or feel that something is amiss, leading them not to connect their formal definitions to mental images, is called a cognitive conflict (D. Tall, 1988; D. O. Tall & Vinner, 1981).

Edwards and Ward (2008) and Torkildsen et al. (2023) reviewed previous work in mathematics, mathematics education, philosophy, and lexicography and proposed a framework for creating and using mathematical definitions. In particular, they distinguished between *extracted* definitions and *stipulated* definitions.

> *Extracted* definitions report usage, while *stipulated* definitions create usage, indeed create concepts, by decree Edwards and Ward (2008, p. 224)
> *Extracted* definitions explain how a word is used by people through examples and can be thought of as a descriptive definition. Meanwhile, *stipulated* definitions are formal and rigorous and determine the one and only meaning of a term. (Torkildsen et al., 2023, p. 5-8)

Extracted and *stipulated* definitions connect to a study of De Villiers (2009) related to descriptive defining and constructive defining.

> *Descriptive* (a posteriori) defining of a concept is meant here that the concept and its properties have already been known for some time and is defined only afterwards. (De Villiers, 2009, p. 2)
> *Constructive* (a priori) defining takes places when a given definition of a concept is changed through the exclusion, generalisation, specialisation, replacement, or addition of properties to the definition. (De Villiers, 2009, p. 2)

Torkildsen et al. (2023) conducted a literature review and found that the above two terms, *stipulated* and *constructed*, and another term, *arbitrary*, are the intrinsic properties of mathematical definitions. Torkildsen et al. (2023) defined "*arbitrary* as the choice of statement to use as a definition for the concept" (p. 7). For example, students are asked to choose a definition of an even number from these two definitions:

(1) An even number is a number that can be divided into two equal groups.
(2) An even number is a number for which the last digit is 0, 2, 4, 6, or 8.

These two equivalent definitions will produce different arguments from students. As a result, students can learn to choose a definition for a purpose.

Methods and materials

Participants

Participants were third-year undergraduate mathematics students at a leading university in the United Kingdom. This group was selected purposely since, compared to those in earlier years, third-year students often possess a better grasp of fundamental mathematical concepts as well as analytical and problem-solving abilities due to their increased experience in solving complex mathematical tasks. Most students in this group have also taken courses in linear algebra and calculus, which require understanding and practical use of factoring concepts. Thus, we believed that they could help answer our research questions.

Students were invited to take part in the study and given the participant information sheet before participating voluntarily in this research. In total, seventeen students agreed to participate, and we have labelled their responses as S1, S2, S3, S4, S5, S6, S7, S8, S9, S10, S11, S12, S13, S14, S15, S16, and S17.

Instruments

The questionnaire included open-ended and closed-ended questions. Open-ended questions allowed for diverse responses and enabled researchers to gain richer insights into participants' perspectives and experiences on factored forms. Closed-ended questions using yes-or-no questions could objectively and specifically assess students' comprehension of factorisation concepts, stimulate critical thinking by challenging them to justify their answers and lead to a discussion if students answered with diverse responses. The questions were divided into four parts. The first part asked students to define a factored form. Questions were developed to first ask students' definitions of a factored form to prevent the possibility of disrupting students' abilities to give clear statements of definition. Students were also asked to redefine the definition in the last task to uncover possible changes and inconsistencies. The other three parts asked students to: (i) identify whether an algebraic expression is a factored form and explain the reason behind that; (ii) investigate whether an algebraic expression is factored over different number systems; and (iii) identify a factored form in terms of powers and distinct terms.

Data Analysis Technique

The data were analysed using both qualitative and quantitative methods. A thematic analysis was used to identify recurring themes or patterns based on key concepts, ideas, or characteristics related to factored forms. The six steps of thematic analysis are familiarising with the data, generating initial codes, searching for themes, reviewing themes, defining and naming themes, and producing the report (Braun & Clarke, 2006). Questions 1 and 5 in the questionnaire focused on asking students' definitions of a factored form. The themes here were coded using existing previous research: well-defined (formally defined), ill-defined or vague, and other valid definitions (Wladis et al., 2022), minimal definition, and insufficient definition (De Villiers, 2009; Zaslavsky & Shir, 2005). We judged a definition to be *well-defined* when it was expressed in what we considered to be a mathematically rigorous and precise manner that allowed for clear and unambiguous application in various mathematical contexts. *A minimal definition* is a concise and clear statement that only captures the essential properties or characteristics necessary to identify a concept without unnecessary details. *An insufficient definition* is one that we judge lacks essential properties or needs to be more complex, making it imprecise and difficult to understand or apply effectively. The themes obtained were analysed using descriptive statistics. Students' responses to question 2 were analysed using descriptive statistics to determine the number of students who answered "Yes" or "No" when identifying a factored form. Meanwhile, the students' reasons behind their answers were coded using thematic analysis. Students' responses to questions 3 and 4 were also analysed using thematic analysis and descriptive statistics.

Results

Students' definitions of a factored form

After students were asked Questions 1 and 5 to define a factored form, no differences were found in students' responses to the two questions.

Based on the definition's accuracy, clarity and adequacy, students gave three kinds of definitions, as shown in Table 1.

Five students provided well-defined definitions, formally articulating standard and accepted definitions, as observed in well-articulated explanations (Wladis et al., 2022). They defined a factored form as a product of factors that cannot be further reduced. For instance, S3, S8, S14, and S16 provided descriptive definitions as follows:

> Factored polynomial is a polynomial which is written as a product of non-constant polynomials, which cannot be factored further. (S3)

> A polynomial is in a factored form if and only if it is written as the product of polynomials, each of the lowest integer degree possible (irreducible). (S8)
> The factored form separates an algebraic expression into the product of expressions, and expressions are irreducible. (S14)
> The “factored form” is a reduced form where the polynomial is rewritten as the product of polynomials of smaller degrees, which themselves cannot be written as the product of other polynomials of smaller degrees. (S6)

Meanwhile, S7 employed symbolic notation to provide a precise definition of a factored form as follows:

> The factored form of an algebraic expression A consists of a product (potentially infinite) of the form $A = N\,(M_1)^{q_1}\,(M_2)^{q_2} \ldots (M_n)^{q_n}$, where each M_i is a unique factor of the algebraic expression, i.e. we have that for all i, M_i divides A. N is some integer, and each q_i is also an integer. (S7)

Eight students held a minimal definition of a factored form, providing a clear definition but only capturing one essential property of factored forms (Zaslavsky & Shir, 2005). S2, S4, and S11 defined a factored form as a product of all factors without further elaborating on the details related to irreducibility. Their explanations are outlined as follows:

> The expression written as a product of all its factors. (S2)
> ...when a polynomial has its real roots factored out and written as a product of its roots. (S4)
> To group common terms together..., resulting in a chain of multiplied terms. (S11)

We also include student S1 in the minimal definition group.

> An algebraic expression is written in a factored form if it is written as the product of the variable minus its root. Slightly more explicitly, a polynomial is in a factored form if $p(x) = a_0(a_1x-r_1)(a_2x-r_2) \ldots (a_2x-r_n)$ for constants $a_0 \ldots n$ and $r_1 \ldots n$. (S1)

This student has not explicitly discussed irreducibly, but they have explicitly used linear factors which were irreducible.

S5, S9, S12, and S15 articulated a factored form as an irreducible form, yet they did not provide further elaboration on how irreducible factors were represented as a product of factors, as demonstrated below:

> When the expression is factorised as much as possible and cannot be further factorised. (S5) A polynomial is written in terms of its factors, all of which are individually irreducible. (S9) The part in the brackets cannot be factored any more. (S12)
> An expression where we cannot extract any more common factors from terms. (S15)

Four students presented definitions that we judged insufficient, incompletely defining a factored form as follows:

> A common multiple of all the terms. (S16)
> (ax+b)(cx+d) for a polynomial to the power of 2. (S13)
> They also offered imprecise definitions, as exemplified by S10 and S17.
> An expression simplified to its simplest form. (S10)
> When you've factored out all the common terms. (S17)

The insufficient definitions lacked adequate information to ensure acceptance by others (Zaslavsky & Shir, 2005) and to facilitate understanding of the concept being discussed (De Villiers, 2009).

Students' identification of factored forms

After providing their definitions, students were offered ten algebraic expressions and asked to choose whether these were factored or not in Question 2. The results are presented in Table 2.

Based on Table 2, all students identified that $(x+1)(x^2+3x+2)$ was not in a factored form since it could be further factored to $(x+2)(x+1)^2$. They also recognised $(x+1)(x-2)^2$ and $(4+x)^3$ as factored forms, noting that these expressions were irreducible. This suggests that students were familiar with these types of questions. Meanwhile, the groups providing different definitions, as shown in Table 1, had varying responses when identifying the other seven expressions. We structure this discussion by grouping students into those with well-defined definitions, minimal definitions, and insufficient or imprecise definitions.

A group of five students (S3, S6, S7, S8, and S14) who provided well-defined definitions claimed seven other expressions and explained their reasons as in Table 3. Based on students' responses in Table 3, three critical findings were found. *First*, for S6, S7 and S8, extracting an integer factor from a linear expression constituted factoring, while this criterion did not apply for S3 and S14. *Second*, S3 and S8 considered the irreducibility of $(x+5)(x^2-7)$ in the context of a larger (unspecified) coefficient-ring such as $\mathbb{R}$. *Third*, although S6 and S7 did not explicitly state the terms "powers" and "distinct" in the definition, this reasoning indicated that these two students believed that a factored form should utilise the powers of distinct terms.

A group of eight students provided minimal definitions. S1, S2, S4, and S11 defined a factored form as a product of its factors. S1, S2, S4, and S11 responded differently to the six expressions and explained their reasons as in Table 4. The table shows four noticeable results. *First*, S1 considered $9x-3$ as a factored form not due to irreducibility. *Second*, S1 also stated that $(x+2)(x^2+2x+10)$ was not in a factored form, suggesting it could potentially be simplified further since the quadratic expression was not neatly factored. However, S1 did not explain further what the factors were. *Third*, S1 noted that x^2-7 was not linear, indicating a focus on the degree of the variable x of the expressions, not the irreducibility of the expressions. *Fourth*, S4 was the only student who believed that a factored form should utilise the powers of distinct terms.

In contrast, the other four students in the minimal definition group, S5, S9, S12, and S15, defined a factored form as an irreducible form. They responded differently to the six expressions and explained their reasons as in Table 5. Three important results were found in the table. *First*, S12 thought that $x(2x-1)+5(2x-1)$ was in a factored form because $2x-1$ could not be factored further within the real numbers. *Second*, S5 identified $(x+5)(x^2-7)$ as not a factored form, asserting it could be further factorised, although S5 did not specify the factors. *Third*, in terms of "powers" and "distinct" in the factored form, S9, S12, and S15 consistently believed these terms were not important. S5 was the only student who stated that $(x-1)(x-1)(x+1)$ was not a factored form. It is possibly S5 thought (erroneously) that $(x-1)(x+1) = (x-1)$ and then substituted the last two terms in the product $(x-1)(x-1)(x+1)$ producing $(x-1)^2$. The reasoning behind this (outlying) answer requires further investigation to determine whether it stemmed from a misunderstanding or a procedural mistake.

A group of four students, who were S10, S13, S16, and S17, provided a definition we considered insufficient They identified differently to the seven expressions and explained their reasons as in Table 6. Three essential results were found. *First*, S10 identified the expressions as factored forms because all variables x were to the first power. This student's answer indicated a possible conflict between the student's personal definitions and mental representations of a factored form. *Second*, S17 considered $(x+5)(x^2-7)$ was in a factored form, reasoning that it could not be factored further as it was irreducible over Z. S16 precisely indicated that it could be factored as $(x-\sqrt{7})(x+\sqrt{7})$, whereas S13 incorrectly stated it as $(x-7)(x+7)$. *Third*, regarding the powers of distinct terms, S13, S16, and S17 believed these terms were crucial when factoring.

Students' understanding of a factored form

Based on four terms of the definition of a factored form, which are "product", "irreducible", "power" and "distinct" (Sangwin, 2013), students gave definitions as in Table 7.

From Table 7, it could be observed that most students defined *a factored form* as a product of irreducible expressions. However, no student used the powers of distinct terms to define a factored form. In the next section, the extent to which students understand the definition of a factored form in terms of product, irreducible, power, and distinct will be identified.

Product

A slight majority of participants defined *a factored form* as a product of factors (see Table 7), it could be seen when they were asked to identify $9x-3$ and $x(2x-1)+5(2x-1)$. Twelve students stated that $9x-3$ was not factored because it could be rewritten as a product of 3 and $3x-1$ (see Table 2). They considered taking out an integer factor from a linear expression

as factoring. Fourteen students also identified $x(2x-1)+5(2x-1)$ as not factored because it was not written as a product. They considered it should be factored into $(x+5)(2x-1)$. Thus, more than 50% of participants believed that the term product was an essential part of the definition of a factored form.

Irreducibility

Although only nine students defined *a factored form* as a form that could not be reduced further (irreducible), we believed most students understood that a factored form was written as a product of irreducible factors. Most students used the term *irreducible or could not be simplified further* as their reasons when they identified $(x+2)(x^2+2x+10)$, $(x+5)(x^2-7)$, and $(x+1)(x^2+3x+2)$ as being factored. They had different answers about whether the expressions were factored forms, according to their understanding of the fields or number systems that could be used in factoring. Nine students (S2, S6, S7, S9, S11, S12, S14, S15, and S17) stated that $(x+5)(x^2-7)$ was factored because it was irreducible over $\mathbb{Z}$. Meanwhile, others thought that it was not factored because it was not irreducible over another ring, i.e. $\mathbb{R}$.

When students were asked further questions related to the *irreducible* in Question 3, there was a change in their thoughts. S11 and S12, who initially stated that $(x+5)(x^2-7)$ was a factored form, identified that x^2-3 was not factored. They stated that x^2-3 was not factored because it could be reduced to $(x-\sqrt{3})(x+\sqrt{3})$. This result showed they changed their thoughts about the fields or number systems they were working on from $\mathbb{Z}$ to $\mathbb{R}$. The other eight students (S1, S3, S4, S5, S8, S10, S13, and S16) had the same opinion that both $(x+5)(x^2-7)$ and x^2-3 were not factored because they were irreducible over $\mathbb{R}$.

S2, S6, S14, and S17 showed inconsistent responses. They stated that both x^2-3 and $(x-\sqrt{3})(x+\sqrt{3})$were factored, because both expressions were irreducible when working in different fields. They believed the answers depended on the field in which we worked. S7, S9, and S15 had consistent beliefs that both $(x+5)(x^2-7)$ and x^2-3 were only factored over $\mathbb{Z}$.

Powers of Distinct Terms

All students stated that $(x+1)(x-2)^2$ and $(4+x)^3$ were factored, but they had different thoughts regarding $(x+3)(x+3)^2$, $(x-1)(x-1)(x+1)$, and $(x+1)(x+1)$ (see Table 2). Nine students (S1, S2, S3, S8, S9, S10, S11, S12, and S14) believed they were factored, whereas the other seven students (S4, S6, S7, S13, S15, S16, and S17) identified that these three expressions were not in factored forms. S5 stated $(x+3)(x+3)^2$ and $(x+1)(x+1)$ were factored, but this student thought that $(x-1)(x-1)(x+1)$ was not factored. This student's response (see Table 5) suggests this student may have made an algebraic mistake.

When students were asked further questions regarding the powers of distinct terms in Question 4, fourteen students (S1, S2, S3, S4, S7, S8, S9, S10, S11, S12, S13, S14, S15, and S17) provided consistent responses. Nine students (S1, S2, S3, S8, S9, S10, S11, S12, and S14) consistently answered that the terms "power" and "distinct" were not crucial in factoring. The other five students also responded consistently that the powers of distinct terms were important. They thought an expression should be written as the powers of different expressions if it was factored.

In contrast, the other two students' responses, S6 and S16, were entirely changed. They, who initially believed powers of distinct terms were important, stated that both $(2z+1)(2z-1)^2$ and $(2z+1)(2z-1)(2z-1)$ were factored. They thought that these two expressions were similar and could not be factored further as long as all expressions could not be reduced further. The nine students had the same thought as S6 and S16 did. However, S6 and S16 stated that $(2z+1)(2z-1)^2$ was a better notation and more simplified. S5 was the only student who stated that $(2z+1)(2z-1)(2z-1)$ was factored but $(2z+1)(2z-1)^2$ was not factored. This student appears to believe that a factored form should explicitly show each of its factors without powers. Thus, most students believed that "the powers of distinct terms" was not pivotal for factoring. The insignificance of the powers of distinct terms is also shown in Table 7.

Discussions

Although we paraphrased Question 1 “Please define the factored form of an algebraic expression, such as a polynomial!” to Question 5 “What does a factored form mean?”, all students stated the exact definition both before and after identifying factored forms. This finding showed that the students believed that defining a concept is the same as understanding what a concept means. We are asking students first to define a concept because we want to check whether students know the definition of a factored form and can use it to identify a factored form. Meanwhile, we want to analyse whether students would change and correct their definitions after classifying or reasoning tasks. In this study, the classifying or reasoning tasks could not be associated with their definitions. However, students used their previous definitions when identifying a factored form. Therefore, this study supports previous researchers Alcock and Simpson (2017) and Inglis and Simpson (2008), who found that giving a defining task first could encourage students to understand better a classification or reasoning task given next.

In defining a concept of a factored form, we judged those students used their concept definitions where they stated verbal explanations clearly and accurately (Vinner, 1983). However, their definitions could be categorised into two broad types: *formal*, widely accepted and recognised, and *personal* definitions stemming from individual understanding (D. O. Tall & Vinner, 1981). We judged that *a well-defined* definition could represent a formal definition, aligning with the definition of a factored form in mathematics textbooks (see section 2) as “a product of irreducible forms”. They defined a factored form using the terms: written as a product; the product of expressions; a product of its roots; multiplied terms; cannot be further factorised; cannot be factorised anymore; irreducible; smaller degrees; lowest integer degree; the simplest form; and cannot extract anymore. Meanwhile, students who stated *minimal* and *insufficient* definitions might reflect their personal concept definition. They had yet to capture well-accepted definitions because they might rely more on concept images influenced heavily by cognitive understanding and personal experiences (Biza et al., 2008; D. O. Tall & Vinner, 1981). This result showed that most students might forget the formal definition of a factored form but could state their definition constructed from their evoked concept images (Alyami, 2023). An evoked concept image means "the portion of the concept image which is activated at a particular time" (D. O. Tall & Vinner, 1981, p. 152).

Regarding the use of words when defining a factored form, most students provided *extracted* definitions (Edwards & Ward, 2008; Torkildsen et al., 2023) or applied *descriptive* defining (De Villiers, 2009). They applied words commonly understood by people. For example, a student defined a factored form as "an expression simplified to its simplest form". This descriptive definition could be categorised into a provoked concept definition defined as "a personal concept definition produced when a mathematical problem elicits or encourages describing a concept" (Alyami, 2023, p. 161).

On the other hand, a student, S7, constructed a definition of a factored form that De Villiers (2009) called *constructive* defining. S7 explained every term used in more detail and provided some examples of factored forms: $x^2+1 = (x+i)(x-i)$ and $x^2-1 = (x+1)(x-1)$. S7 also provided a well-defined definition that we judged formal and rigorous. Thus, S7 has stated a *stipulated* definition (Edwards & Ward, 2008; Torkildsen et al., 2023). S7 was able to understand two intrinsic properties of mathematical definitions, *stipulated* and *constructed* (Torkildsen et al., 2023).

A group of five students who provided *well-defined* definitions using the terms "product" and "irreducibility" clearly applied these terms when identifying a factored form. They thoroughly understood that a factored form is a product of its roots. However, they differed in applying the terms of irreducibility. Three students in this group thought they were working over $\mathbb{Z}$, while others believed it should be in $\mathbb{R}$. These students did not state clearly the terms “powers” and “distinct” in their definitions. However, two of them believed that these terms were important in factoring. This perspective likely stemmed from the students’ non-verbal knowledge or mental representations of the concept of a factored form. Vinner (1991) referred to these mental representations as concept images developed through personal experiences.

Most students of the *minimal-definition* group identified the expressions with clear reasoning, similar to the *well-defined* group. Even though S1, S2, S4, and S11 defined a factored form solely as a product, they could explain that an expression was not in a factored form if it could be factored or reduced further. S1 understood a factored form as a product, as stated in S1’s definition. However, this student had a vague understanding related to “irreducibility”. This understanding may have occurred due to imprecise ideas within the student’s concept image or possible cognitive conflicts between the student’s definition and the concept images (D. O. Tall & Vinner, 1981). S5, S9, S12, and S15, who defined a factored form as an irreducible expression, believed the expression should be written as a product if it was factored. Thus, these eight students could identify the expressions using their concept definitions and their concept images or non-verbal representations. According to Vinner (1983), when students can link their concept definitions to their concept images, they develop a solid conceptual understanding. The results, including their answers and explanations, align with Vinner’s statement.

Although students in the *insufficient* definition group provided vague and ambiguous definitions, most demonstrated the ability to identify a factored form by determining whether the expression was a product or irreducible. This understanding likely stemmed from their knowledge and experiences, referred to as concept images. Most students had distinct reasons for their answers, reflecting the variability in individual reactions to a specific concept based on their concept images (Vinner, 1991). This result aligns with the findings of Li and Tall (1993). They found that students who were first-year trainees in mathematics at Warwick University in a course on programming and numerical methods did not depend on the concept definition for mathematical tasks; instead, they drew upon a concept image derived from their own experiences.

The results of this study clearly show that students' conceptual understanding was formed by their concept definitions and concept images. Concept definitions can provide clear and precise foundations, while concept images can lead students to reach a comprehensive understanding. Vinner (1991, p. 69) believed that "to understand means to have a concept image". A concept definition helps to form a concept image. Vinner (1983) have also explained the models of forming concept definitions and concept images. These models encourage us to consider what should come and taught first.

Some students thought that taking out a numerical factor from a linear expression was factoring. They recognised $9x-3$ shared a common factor of 3 and then factorised it into $3(3x-1)$. Meanwhile, other students believed $9x-3$ was irreducible. These students' reasons should be investigated whether they believed factoring only involves taking out common monomial factors. However, Hall and Knight (1962) stated that factoring polynomials proceeds by taking out numerical and common monomial factors. Regarding a factored form as a product of irreducible terms, S10, S12, and S16 understand that $x(2x-1)+5(2x-1)$ is irreducible. These students do not consider that a factored form must be a product of its factors. S10 understands that an expression is irreducible if each variable x is to the first power. This student's understanding has to be clarified. This student's misunderstanding of the meaning of irreducible led him to identify a factored form erroneously. Although no students used "powers of distinct terms" to define a factored form, some students considered it important in factoring an expression. They believed that a factored form should be written in the form of powers of distinct terms. Therefore, most students are familiar with the definition of a factored form as a product of irreducible terms, as explained in many mathematics textbooks (see section 2). However, each student's concept image causes students differ in understanding the use of powers of distinct terms.

Conclusions

Most students did not primarily rely on their concept definitions when tackling mathematical questions. Instead, they tended to favour their conceptual images. However, students who possessed well-defined definitions exhibited a robust conceptual understanding. They demonstrated clarity in identifying concepts and the rationale behind their answers. Moreover, they adeptly applied their conceptual images and effectively linked them to their concept definitions while solving problems, resulting in minimal interpretation conflicts. Thus, both concept definitions and concept images had crucial roles in forming students' understanding.

Further research is required to deepen our understanding of how students apply definitions when solving mathematical problems. Specifically, future studies should examine algebraic expressions using negative signs for x^n, for example, x^2-3x-2 or x^3+1, and those over the rational ($\mathbb{Q}$) and complex ($\mathbb{C}$) number systems, to identify undergraduate students' comprehension of factoring across different fields. Additionally, using interviews to investigate students' mathematical thinking and the reasons behind their definitions will be interesting. It would also offer valuable insights into the connection between concept definitions and conceptual images. In this study, data collection was limited to a questionnaire. Students tended to provide concise responses when questioned via a questionnaire, which showed that further verification using an interview was required to obtain more comprehensive insights

Acknowledgments

The authors would like to express the deepest gratitude to the Indonesia Endowment Fund for Education (LPDP) under the Ministry of Finance of the Republic of Indonesia for funding the author's PhD study.

Declaration of interest

The authors report there are no competing interests to declare.

Ethics approval

Ethics approval was granted by the University of Edinburgh. Participant information sheet and consent to participate were freely given, and participation in this study was voluntary.

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Table 1. The accuracy, clarity and adequacy of students' definitions.

Definition Type	Characteristics	Students	%
Well-defined definition	Clear, precise, and accurate definitions encompassing both product and irreducibility concepts	S3, S6, S7, S8, S14	29%
Minimal definition	Basic definitions focusing on either product or irreducibility but with some inconsistencies	S1, S2, S4, S5, S9, S11, S12, S15	47%
Insufficient definition	Vague or incomplete definitions, lacking clarity on key aspects of factored forms	S10, S13, S16, S17	24%

Table 2. Students' identification of algebraic expressions as factored or not.

No	Expressions	Answers and Initials			
		Y	Students	N	Students
1	$9x-3$	5	S1, S3, S9, S10, S14	12	S2, S4, S5, S6, S7, S8, S11, S12, S13, S15, S16, S17
2	$x(2x-1)+5(2x-1)$	3	S10, S12, S16	14	S1, S2, S3, S4, S5, S6, S7, S8, S9, S11, S13, S14, S15, S17
3	$x^2+2x+10$	15	S2, S3, S4, S5, S6, S7, S8, S9, S11, S12, S13, S14, S15, S16, S17	2	S1, S10
4	$(x+5)(x^2-7)$	9	S2, S6, S7, S9, S11, S12, S14, S15, S17	8	S1, S3, S4, S5, S8, S10, S13, S16
5	$(x+1)(x^2+3x+2)$	0	-	17	1, S2, S3, S4, S5, S6, S7, S8, S9, S10, S11, S12, S13, S14, S15, S16, S17
6	$(x+3)(x+3)^2$	10	S1, S2, S3, S5, S8, S9, S10, S11, S12, S14	7	S4, S6, S7, S13, S15, S16, S17
7	$(x+1)(x-2)^2$	17	S1, S2, S3, S4, S5, S6, S7, S8, S9, S10, S11, S12, S13, S14, S15, S16, S17	0	-
8	$(x-1)(x-)(x+1)$	9	S1, S2, S3, S8, S9, S10, S11, S12, S14	8	S4, S5, S6, S7, S13, S15, S16, S17

9	$(x+1)(x+1)$	10	S1, S2, S3, S5, S8, S9, S10, S11, S12, S14	7	S4, S6, S7, S13, S15, S16, S17
10	$(4+x)^3$	17	S1, S2, S3, S4, S5, S6, S7, S8, S9, S10, S11, S12, S13, S14, S15, S16, S17	0	-

Table 3. The responses of well-defined definition group.

Expressions	Answers and Initials			
	Y	Students	N	Students
$9x-3$	S3, S14	Irreducible	S6, S7, S8	It could be transformed to $3(3x-1)$
$x(2x-1)+5(2x-1)$	-	-	S3, S6, S7, S8, S14	It was not a product
$x^2+2x+10$	S3, S6, S7, S8, S14	Irreducible	-	-
$(x+5)(x^2-7)$	S6, S7, S14	Irreducible	S3, S8	It could be factored as $(x-\sqrt{7})(x+\sqrt{7})$,
$(x+3)(x+3)^2$ $(x-1)(x-1)(x+1)$ $(x+1)(x+1)$	S3, S8, S14	Irreducible	S6, S7	It could be written using unique factors: $(x+3)^3$, $(x-1)^2(x+1)$, $(x+1)^2$

Table 4. The responses of a group of students who defined a factored form as a product.

Expressions	Answers and Initials			
	Y	Students	N	Students
$9x-3$	S1	It was a product because it included a scalar factor	S2, S4, S11	It could be transformed to $3(3x-1)$
$x(2x-1)+5(2x-1)$	-	-	S1, S2, S4, S11	It could be factored to $(2x-1)(x+5)$
$x^2+2x+10$	S2, S4, S11	Irreducible	S1	Reducible
$(x+5)(x^2-7)$	S2, S11	Irreducible	S1, S4	S1: Reducible S4: It could be written as a difference of two squared
$(x+3)(x+3)^2$ $(x-1)(x-1)(x+1)$ $(x+1)(x+1)$	S1, S2, S11	Irreducible	S4	It could be written into: $(x+3)^3$, $(x-1)^2(x+1)$, $(x+1)^2$

Table 5. The responses of a group of students who defined a factored form as an irreducible form.

Expressions	Answers and Initials			
	Y	Students	N	Students
$9x-3$	S9	Irreducible	S5, S12, S15	It could be transformed to $3(3x-1)$
$x(2x-1)+5(2x-1)$	S12	Irreducible	S5, S9, S15	It could be expressed as a product: $(2x-1)(x+5)$
$x^2+2x+10$	S5, S9, S12, S15	Irreducible	-	-
$(x+5)(x^2-7)$	S9, S12, S15	Irreducible	S5	Reducible
$(x+3)(x+3)^2$	S5, S9, S12, S15	Irreducible	-	-
$(x-1)(x-1)(x+1)$	S9, S12, S15	Irreducible	S5	$(x-1)(x+1) = (x-1)$. Thus, the expression is $(x-1)^2$[sic]
$(x+1)(x+1)$	S5, S9, S12, S15	Irreducible	-	-

Table 6. The responses of insufficient definition group.

Expressions	Answers and Initials			
	Y	Students	N	Students
$9x-3$	S10	x was to the first power	S13, S16, S17	It could be transformed to $3(3x-1)$
$x(2x-1)+5(2x-1)$	S10, S16	S10: All variables x were to the first power S16: $2x$ and 1 have no common divisor except for 1	S13, S17	It could be written as a product: $(2x-1)(x+5)$
$x^2+2x+10$	S13, S16, S17	Irreducible	S10	Reducible
$(x+5)(x^2-7)$	S17	Irreducible	S10, S13, S16	S10: Reducible S13: $(x+7)(x-7)$ S16: It could be written as the

				difference of two squares
$(x+3)(x+3)^2$ $(x-1)(x-1)(x+1)$ $(x+1)(x+1)$	S10	All variables x were to the first power	S13, S16, S17	It could be written using unique factors: $(x+3)^3$, $(x-1)^2(x+1)$, $(x+1)^2$

Table 7. Students' definitions related to four terms of the definition of a factored form.

Students	The definition of a factored form			
	Product	Irreducible	Power	Distinct
S1	Y	N	N	N
S2	Y	N	N	N
S3	Y	Y	N	N
S4	Y	N	N	N
S5	N	Y	N	N
S6	Y	Y	N	N
S7	Y	Y	N	N
S8	Y	Y	N	N
S9	N	Y	N	N
S10	N	N	N	N
S11	Y	N	N	N
S12	N	Y	N	N
S13	N	N	N	N
S14	Y	Y	N	N
S15	N	Y	N	N
S16	Y	N	N	N
S17	N	N	N	N
%	59%	53%	0%	0%

Algebra Concepts

Participant Information Sheet

Introduction:

You are invited to participate in a survey focused on the concepts in elementary a whether to participate, it is important for you to understand why the research is b your participation will involve.

Purpose of the Study:

The purposes of this study are to explore your understanding of particular conce short sequence of structured questions.

Participant Eligibility:

You are eligible to participate in this study because you are a Year 3 student in th There are no specific other qualifications.

Study Procedures:

Participation in this study involves completing an online survey. It is estimated to minutes to complete.

Potential Risks:

There are minimal risks associated with participating in this study. The questio invasive and pose no physical or psychological harm.

Benefits of Participation:

Your participation will contribute to the advancement of knowledge in the field c Your insights and perspectives will help researchers better understand how indivic with algebra concepts, which could potentially lead to improvements in educationa

Voluntary Participation:

Participation in this study is entirely voluntary. You may choose not to participate c at any time without penalty or consequences. Your decision will not affect your cu with the researchers or their affiliated institutions.

Data Protection and Confidentiality:

Your data will be processed in accordance with Data Protection Law. All informatic be kept strictly confidential. Before we analyse the survey responses, we will rep (name and UUN) with an anonymised identifier. The anonymised data will be he Edinburgh systems during our analysis. We will publish the anonymised data onlin our analysis, so that other researchers can check our analyses and use the data for

Contact Information:

If you have any questions or concerns about the study, you may contact the M.Aziza@sms.ed.ac.uk.

If you wish to make a complaint about the study, please contact Professor J Research, School of Mathematics, j.pearson@ed.ac.uk). In your communication, title ("Investigating insight and rigour in mathematical proof") and detail the nature

For general information about how we use your data go to:

https://www.ed.ac.uk/records-management/privacy-notice-research

Participant Statements

By participating in this study,

o I agree to provide honest and accurate information to the best of my ability.
o I confirm that I have read and understood the Participant Information Sheet for t
o I acknowledge that my participation is voluntary and that I have the option to wi any time without penalty or consequences.
o I also understand that my responses will remain confidential and will be used sol

By selecting "I agree", you are consenting to the conditions described a

- ◯ I agree
- ◯ I disagree

Please define the "factored form" of an algebraic expression, such as a p

Is 9x-3 written in a factored form? *

- ◯ Yes
- ◯ No

Why? *

Is x(2x-1)+5(2x-1) written in a factored form? *

Yes

No

Why?

Is $(x+2)(x^2+2x+10)$ written in a factored form? *

Yes

No

Why? *

Is $(x+5)(x^2-7)$ written in a factored form? *

Yes

No

Why? *

Is $(x+1)(x^2+3x+2)$ written in a factored form? *

- ◯ Yes
- ◯ No

Why? *

Is $(x+3)(x+3)^2$ written in a factored form? *

◯ Yes

◯ No

Why? *

Is $(x+1)(x-2)^2$ written in a factored form? *

◯ Yes

◯ No

Why? *

Is (x-1)(x-1)(x+1) written in a factored form? *

- Yes
- No

Why? *

Is (x+1)(x+1) written in a factored form? *

◯ Yes

◯ No

Why? *

Is $(4+x)^3$ written in a factored form? *

◯ Yes

◯ No

Why? *

Is x^2-3 factored? *

- ◯ Yes
- ◯ No

Why is x^2-3 not factored? *

Is $(x-\sqrt{3})(x+\sqrt{3})$ factored? *

- ◯ Yes
- ◯ No

Please expand out $(x-\sqrt{3})(x+\sqrt{3})$! You claim both x^2-3 and $(x-\sqrt{3})(x+\sqrt{3})$ are factored. Why is this the case? *

Is (2z+1)(2z-1)(2z-1) factored? *

- ◯ Yes
- ◯ No

Why is (2z+1)(2z-1)(2z-1) not factored? *

Is $(2z+1)(2z-1)^2$ factored? *

- Yes
- No

You claim both (2z+1)(2z-1)(2z-1) and $(2z+1)(2z-1)^2$ are factored. Why is

What does a factored form mean? *